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DIFFERENTIABILITY OF CONVEX FUNCTIONS AND THE CONVEX POINT OF CONTINUITY PROPERTY IN BANACH SPACES

BY

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ABSTRACT

We show that if X is a separable Banach space, then every continuous, convex, Gâteaux differentiable function on X is Fréchet differentiable on a dense set if and only if X^* has the *weak*-Convex Point of Continuity Property* (C^*PCP).

Introduction

The following theorem, which was the culmination of several years of effort by many mathematicians, is now well known, and was the motivation for the results we present in this note:

THEOREM [5]. *A Banach space X is an Asplund space (every equivalent norm is Fréchet differentiable on a dense set) if and only if X^* has RNP (every bounded set is dentable). X has RNP if and only if X^* is a weak*-Asplund space (every equivalent dual norm is Fréchet differentiable on a dense set).*

A Banach space X has the *Convex Point-of-Continuity Property* (*CPCP*) if

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every closed, bounded, convex subset C of X has a point at which the relative weak and norm topologies on C coincide. This property was introduced by Bourgain in [2] (he called it property (*)). Dually, X^* has the *weak*-Point of Continuity Property* (*C*PCP*) if every weak*-compact, convex subset C of X^* has a point at which the relative weak* and norm topologies coincide.

It is immediate that *RNP* implies *CPCP*. The converse is not true, even for spaces with separable dual. A counterexample is the predual of the *James Tree space* ([4], [15]; see also [10]).

It follows from the results of Godefroy and Maurey [13] that if X is a separable Banach space which contains an isomorphic copy of l_1, then X^* does not have *C*PCP*. On the other hand, it is shown in [12] that the dual of the James Tree space has *C*PCP*, and that there is a Banach space which does not contain l_1 and yet X^* does not have *C*PCP*.

The following result provides dual characterizations for *CPCP* and *C*PCP*:

THEOREM 1. *Let X be a separable Banach space. Then*:

(a) *Every equivalent norm on X with strictly convex dual is Fréchet differentiable on a dense set if and only if X* has C*PCP.*

(b) *The dual of every equivalent strictly convex norm on X is Fréchet differentiable on a dense set if and only if X has CPCP.*

In proving this Theorem we also obtain the following analogue of the *Bishop–Phelps Theorem* (see Proposition 7): If X has *CPCP* and $C \subset X$ is closed, bounded and convex, then the set of functionals which attain their supremum over C at a point of weak-to-norm continuity of C is norm dense in X^*. Recall [3] that if C is non-dentable, then the set of supremum attaining functionals on C is of first category in X^*.

By analogy with Asplund space, we will call a Banach space X a *Phelps space* if every continuous, convex, *Gâteaux differentiable* function on X is Fréchet differentiable on a dense set. (In [18], Phelps exhibited a Gâteaux-smooth, nowhere Fréchet differentiable norm on a Banach space, thus answering a question posed by Mazur in [16].) In Section 2, we prove a third equivalence for Theorem 1(a), namely:

THEOREM 2. *Let X be a separable Banach space. Then X is a Phelps space if and only if X* has C*PCP.*

We conclude this note with some remarks concerning the properties of Phelps spaces.

All Banach spaces are assumed to be real. Our notation is standard. In particular, we denote the unit ball of X by $\mathscr{B}_X$, the unit sphere of X by $\mathscr{S}_X$, and the dual on X^* to a norm $\|\cdot\|$ on X by $\|\cdot\|^*$.

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1. Proof of Theorem 1

We recall some definitions:

(1) If $C \subset X$, $f \in X^*$, and $\alpha > 0$, then the set $S(f, C, \alpha) \equiv \{x \in C : f(x) > \sup f(C) - \alpha\}$ is called a *slice* of C.

(2) $x \in C$ is a *denting point* of C if the slices of C containing x form a base for the relative norm topology on C at x.

(3) $x \in C$ is *strongly exposed* if there is an $f \in X^*$ such that if $\{x_n\} \subset C$ and $f(x_n) \to f(x)$, then $x_n \to x$. The functional f is called a *strongly exposing* functional for x. Note that a strongly exposed point is a denting point.

(4) A norm on X is *strictly convex* if every point of norm 1 is an extreme point of the unit ball of X.

LEMMA 3. *Let X be a Banach space with CPCP. Let $C \subset X$ be closed, bounded and convex. Let $f \in X^*$ and $\alpha \in \mathbf{R}$ be such that* $\inf f(C) < \alpha < \sup f(C)$. *Let $x_0 \in f^{-1}(\alpha) \cap C$ be a point of weak-to-norm continuity for $f^{-1}(\alpha) \cap C$. Then x_0 is a point of weak-to-norm continuity for C.*

PROOF. Let $\varepsilon > 0$ and let V be an elementary weakly open neighbourhood of x_0 such that

$$\operatorname{diam} V \cap f^{-1}(\alpha) \cap C < \varepsilon/6.$$

Then we can choose $x_1, x_2 \in V \cap C$ such that

$$f(x_1) - f(x_0) = f(x_0) - f(x_2) \equiv \beta > 0.$$

Let

$$K_i = \{x = x_i + t(y - x_i) : t > 0, y \in V \cap f^{-1}(\alpha) \cap C\},$$

$i = 1, 2$. K_i is the positive cone generated by x_i and $V \cap f^{-1}(\alpha) \cap C$. Notice that if $\alpha - \beta < \gamma < \alpha + \beta$, then

$$(*) \qquad \operatorname{diam} f^{-1}(\gamma) \cap C \cap (K_1 \cup K_2) < \varepsilon/3.$$

Choose x_3, x_4 on the line segment between x_1 and x_2 such that $f(x_3) > \alpha > f(x_4)$ and

$$(**) \qquad \| x_3 - x_4 \| < \varepsilon/3.$$

Let

$$V_1 = V \cap \{x \in X : f(x_3) > f(x) > f(x_4)\}.$$

Then $x_0 \in V_1 \cap C \subset K_1 \cup K_2$. We claim that diam $V_1 \cap C < \varepsilon$. Indeed, let $y, z \in V_1 \cap C$ and denote by y_1 the point of intersection of the hyperplane $f^{-1}(f(y))$ and the line segment $[x_1, x_2]$, and similarly $z_1 = f^{-1}(f(z)) \cap [x_1, x_2]$. Then $y, y_1, z, z_1 \in C \cap (K_1 \cup K_2)$, so by $(*)$, $\| y - y_1 \| < \varepsilon/3$ and $\| z - z_1 \| < \varepsilon/3$. Since $[y_1, y_2] \subset [x_3, x_4]$, we have, by $(**)$, $\| y_1 - z_1 \| < \varepsilon/3$. Thus $\| y - z \| < \varepsilon$, and the proof is complete. ∎

LEMMA 4. *Let X be a Banach space with CPCP, $\varepsilon > 0$, and $f \in X^*$. Let $K \not\subset f^{-1}(0)$ be closed, bounded and convex, and*

$$C = \overline{\text{co}}((f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_X) \cup K).$$

Then there exists an $x \in K$ which is a point of weak-to-norm continuity of C.

PROOF. Without loss of generality, we may suppose that $\sup f(K) > 0$. Choose $0 < \alpha < \sup f(K)$. If the set $C_1 = \text{co}((f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_X) \cup K)$ were closed, then we could obtain a point of weak-to-norm continuity for C_1 in the set $f^{-1}(\alpha) \cap C_1$, by Lemma 3, and then using convexity we could slide this point up to K and be done. As C_1 is not, is general, closed, we transfer the problem to X^{**}, where we can take advantage of compactness.

Thus let $D = \text{co}((f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_{X^{**}}) \cup \bar{K}^*)$, where $\bar{K}^*$ denotes the weak*-closure of K in X^{**}, and we consider f as an element of X^{***}. Since D is exactly the range of the continuous map $(x, y, t) \mapsto (1-t)x + ty$, from $(f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_{X^{**}}) \times \bar{K}^* \times [0, 1]$ to X^{**}, D is weak*-compact. By the weak*-density of K in $\bar{K}^*$ and of $f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_X$ in $f^{-1}(0) \cap \varepsilon^{-1}\beta_{X^{**}}$, we have that $D = \bar{C}^*$.

Let z be a point of weak-to-norm continuity for $f^{-1}(\alpha) \cap C$. By Lemma 3, z is a point of weak-to-norm continuity for C. It follows that z is a point of weak*-to-norm continuity for D. Since $f(z) \neq 0$, there are elements $x \in \bar{K}^*$, $y \in f^{-1}(0) \cap \varepsilon^{-1}\beta_{X^{**}}$, and $0 < t \leqq 1$ such that $z = tx + (1-t)y$. Let $\delta > 0$ and let V be a weak*-open neighbourhood of z such that diam $V \cap D < t\delta$. Then

$$W = \frac{1}{t}V - \frac{1-t}{t}y$$

is a weak*-open neighbourhood of x, and by convexity,

$$W \cap D \subset \frac{1}{t}(V \cap D) - \frac{1-t}{t}y.$$

Thus diam $W \cap D < \delta$. Hence x is a point of weak*-to-norm continuity for D, and x is in $\bar{K}^*$. By the weak*-density of K in $\bar{K}^*$, $x \in K$, and is clearly a point of weak-to-norm continuity for C. ■

Note that if we assume in Lemma 4 that K is the unit ball of a strictly convex norm on X, then the point $x \in K$ produced by the Lemma is, in fact, an extreme point of D, and hence of C. To see this, suppose that $x = (y+z)/2$, where $y, z \in D$. The there exist $y_1, z_1 \in \bar{K}^*$, $y_2, z_2 \in f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_{X^{**}}$ and $0 \leqq \alpha, \beta \leqq 1$ so that $y = \alpha y_1 + (1-\alpha)y_2$ and $z = \beta z_1 + (1-\beta)z_2$. Note that D has non-empty interior, so by the Hahn–Banach theorem, there is a $g \in X^{***}$ such that $g(x) = \sup g(D)$. By convexity,

$$g(x) = g(y) = g(z) = g(y_1) = g(y_2) = g(z_1) = g(z_2).$$

Now, by the same argument as in the proof of Lemma 4, the points y, z, y_1 and z_1 are all points of weak*-to-norm continuity, and hence are in X. Thus, $y_1, z_1 \in K$. Since also $x \in K$, we have, by strict convexity, $y_1 = z_1 = x$. Thus

$$x = \frac{y+z}{2} = \left(\frac{\alpha+\beta}{2}\right)x + \left(\frac{1-\alpha}{2}\right)y_2 + \left(\frac{1-\beta}{2}\right)z_2.$$

Since $f(x) \neq 0$ and $f(y_2) = f(z_2) = 0$, we must have $(\alpha+\beta)/2 = 1$, and so $\alpha = \beta = 1$, that is, $x = y = z$.

LEMMA 5 (Bishop–Phelps [1]). *Let $f, g \in S_{X^*}$ and let $\varepsilon > 0$. If $g(x) = 1$ and $f(x) = 0$ implies $\| x \| > \varepsilon^{-1}$, then $\| f - g \| < 2\varepsilon$ or $\| f + g \| < 2\varepsilon$.*

PROOF OF THEOREM 1. We prove only part (b). The proofs for part (a) are similar.

(b) ($\Leftarrow$) Let $\| \cdot \|$ be an equivalent strictly convex norm on X. For each positive integer n, let

$$U_n = \{ f \in S_{X^*} : \exists\, 0 < \alpha < 1 \text{ such that } \operatorname{diam} S(f, \mathscr{B}_X, \alpha) < n^{-1} \}.$$

We claim that U_n is norm open and dense in the unit sphere of X^*, and hence, by the Baire Category Theorem, $U = \bigcap U_n$ is also dense. Given the claim, it is clear that each $f \in U$ is a strongly exposing functional for X. By Šmuljan's test [7], f is a point of Fréchet differentiability of $\| \cdot \|^*$.

To see that U_n is open, let $f \in U_n$, with corresponding slice $S(f, \mathscr{B}_X, \alpha)$. If $\varepsilon, \beta > 0$ are chosen so that $\beta < \alpha - 2\varepsilon$, and if $g \in \mathscr{S}_{X^*}$ such that $\| f - g \| < \varepsilon$, then it follows easily that $\varnothing \neq S(g, \mathscr{B}_X, \beta) \subset S(f, \mathscr{B}_X, \alpha)$, and so $g \in U_n$.

To see that U_n is dense, let $f \in \mathscr{S}_{X^*}$. Using the notation of Lemma 4 (with $K = \mathscr{B}_X$), there is an $x \in \mathscr{B}_X$ which is a point of weak-to-norm continuity for C. Since $\| \cdot \|$ is strictly convex, the remark following Lemma 4 gives that x is an extreme point of C, and thus, by a result of Lin, Lin and Troyanski [14], x is a denting point of C. It is immediate that if $g \in \mathscr{S}_{X^*}$ and $\alpha > 0$ are such that $\operatorname{diam} S(g, C, \alpha) < n^{-1}$, then $g \in U_n$ and f and g satisfy the hypotheses of Lemma 5. By the symmetry of C, we are done.

($\Rightarrow$) This part of the proof has been motivated by Phelps' example in [18] and Edelstein's example in [9].

Suppose X does not have *CPCP*. Let $\| \cdot \|_1$ denote the original norm on X. Since X is separable, it has a countable norming set $\{ f_j \}_1^\infty$ in the unit sphere of X^*. Since X does not have *CPCP*, there is a non-empty, closed, bounded, convex subset C of X and an $\varepsilon > 0$ such that if U is a weakly open set in X, then $\operatorname{diam} U \cap C \geqq \varepsilon$ if $U \cap C \neq \varnothing$. It is not difficult to observe that the same property holds (with the same ε) for, successively, $-C$, $C + (-C)$, $\mathscr{B}_2 = \mathscr{B}_1 + C + (-C)$, and $\mathscr{B} = \bar{\mathscr{B}}_2$, where $\mathscr{B}_1$ denotes the unit ball of $\| \cdot \|_1$.

Let $|\!|\!| \cdot |\!|\!|$ denote the Minkowski functional of $\mathscr{B}$ and define an equivalent norm $\| \cdot \|$ on X by

$$\| x \|^2 = |\!|\!| x |\!|\!|^2 + \sum_{j=1}^{\infty} 2^{-j} f_j^2(x)$$

for $x \in X$.

We show that $\| \cdot \|$ is a strictly convex norm whose unit ball is not dentable, and hence $\| \cdot \|^*$ cannot be Fréchet differentiable anywhere.

Suppose $x, y \in X$ such that $\| x \| = \| y \| = \| (x + y)/2 \| = 1$. By a standard convexity argument,

$$f_j^2(x) + f_j^2(y) - f_j^2\left(\frac{x+y}{2}\right) = f_j^2\left(\frac{x-y}{2}\right) = 0$$

for all $j = 1, 2, \dots$. By the choice of $\{f_j\}$, $x = y$, and so $\|\cdot\|$ is strictly convex.

Now let

$$m = \inf\{ |||x||| : \|x\| = 1\} \quad \text{and} \quad M = \sup\{ \|x\|_1 : |||x||| < 1\}.$$

Let $f \in X^*$ with $\|f\|^* = 1$, and let $\delta > 0$. Choose an $x_0 \in X$ with $\|x_0\| = 1$ such that $f(x_0) > 1 - \delta$ and choose an integer n_0 such that $M^2 2^{2-n_0} < \delta$. Consider the weakly open set

$$U = \{x \in X : |f(x - x_0)| < \delta \text{ and } |f_j(x - x_0)| < \delta/M, j = 1, 2, \dots, n_0\}.$$

Let $\mathscr{B}_3 = \{x \in X : |||x||| \leqq |||x_0|||\}$. Since $x_0 \in U \cap \mathscr{B}_3$, since $|||x_0||| \geqq m$, and since for any weakly open set V in X we have diam $V \cap \mathscr{B} \geqq \varepsilon$ if $V \cap \mathscr{B} \neq \varnothing$, a simple homothety argument shows that diam $U \cap \mathscr{B}_3 \geqq m\varepsilon$, where diameter is meant with respect to the original norm $\|\cdot\|_1$. Therefore, there is an $x_1 \in \mathscr{B}_3$ such that $\|x_1 - x_0\|_1 > m\varepsilon/4$, $|f(x_1 - x_0)| < \delta$, and $|f_j(x_1 - x_0)| < \delta/M$ for $j = 1, 2, \dots, n_0$. Since $\|\cdot\|_1 \leqq M|||\cdot|||$ and $|||x_0|||^2 + \sum_{j=1}^{\infty} 2^{-j} f_j^2(x_0) = 1$, it follows that

$$\begin{aligned}
\|x_1\|^2 &= |||x_1|||^2 + \sum_{j=1}^{\infty} 2^{-j} f_j^2(x_1) \\
&\leqq |||x_0|||^2 + \sum_{j=1}^{\infty} 2^{-j} f_j^2(x_0) + \sum_{j=1}^{\infty} 2^{-j}(f_j^2(x_1) - f_j^2(x_0)) \\
&= 1 + \sum_{j=1}^{n_0} 2^{-j}(f_j^2(x_1) - f_j^2(x_0)) + \sum_{j=n_0+1}^{\infty} 2^{-j}(f_j^2(x_1) - f_j^2(x_0)) \\
&\leqq 1 + \sum_{j=1}^{n_0} 2^{-j}(|f_j(x_1)| + |f_j(x_0)|)(|f_j(x_1)| - |f_j(x_0)|) \\
&\quad + \sum_{j=n_0+1}^{\infty} 2^{-j}(|f_j(x_1)| + |f_j(x_0)|)^2 \\
&\leqq 1 + (\delta/M) \sum_{j=1}^{n_0} 2^{-j} \|f_j\|_1^*(\|x_1\|_1 + \|x_0\|_1) \\
&\quad + \sum_{j=n_0+1}^{\infty} 2^{-j}(\|f_j\|_1^*(\|x_1\|_1 + \|x_0\|_1))^2
\end{aligned}$$

$$\leqq 1 + (\delta/M)(2M) + 2^{-n_0} \cdot 4M^2$$

$$\leqq 1 + 3\delta.$$

Since δ can be chosen arbitrarily small, it follows that given an $f \in X^*$ with $\| f \|^* = 1$, there are sequences $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$ such that for all $n = 1, 2, \ldots$, $\| x_n \| = \| y_n \| = 1$ and $\| x_n - y_n \|_1 \geqq m\varepsilon/8$, and $\lim f(x_n) = \lim f(y_n) = 1$. Therefore, $\| \cdot \|$ is not dentable. ∎

Recall [8] that a Banach space X has *RNP* if and only if the unit ball of every equivalent norm on X is dentable. Using the result of Lin, Lin and Troyanski mentioned earlier and a standard argument using the Hahn–Banach Theorem and Šmuljan's test, we easily obtain:

COROLLARY 6. *Let X be a separable Banach space. The following are equivalent*:

(a) *X has CPCP.*

(b) *The unit ball of every equivalent strictly convex norm on X is dentable.*

(c) *The unit ball of every equivalent strictly convex norm on X is the closed convex hull of its strongly exposed points.*

We also have the following version of the Bishop–Phelps Theorem:

PROPOSITION 7. *Let X be a Banach space (not necessarily separable) which has CPCP, and let $K \subset X$ be closed, bounded and convex. Then the set of functionals which support K at a point of weak-to-norm continuity of K is norm dense in X^*.*

PROOF. We may assume, without loss of generality, that $K \subset \frac{1}{2}\mathscr{B}_X$. Let $\varepsilon > 0$ and let $f \in \mathscr{S}_{X^*}$. We show that there exists a $g \in \mathscr{S}_{X^*}$ such that $\| f - g \| < 2\varepsilon$ and g supports K at a point of weak-to-norm continuity of K. If f is constant on K, we can clearly choose $g = f$. Otherwise, there exist $u, v, w = (u + v)/2 \in K$ such that $f(u) < f(w) < f(v)$. Let $K_1 = K - w$. Then $K_1 \subset \mathscr{B}_X$ and it suffices to establish the proposition for the set K_1.

By Lemma 4, there is an $x \in K_1$ which is a point of weak-to-norm continuity of $C = \overline{\mathrm{co}}((f^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_X) \cup K_1)$. By the choice of K_1, we may assume $f(x) > 0$. Note that C has non-empty interior. Thus by the Hahn–Banach Theorem, there exists a $g \in \mathscr{S}_{X^*}$ such that $g(x) = \sup g(C) > 0$. Since $K_1 \subset \mathscr{B}_X$, $g(x) \leqq 1$. Thus f and g satisfy the hypotheses of Lemma 5. Since f and g are both positive at x, we must have $\| f - g \| < 2\varepsilon$ if ε is sufficiently small. ∎

2. Convex functions

In this section, we discuss the Fréchet differentiability of Gâteaux-smooth convex functions. Specifically, we prove that separable Phelps spaces are exactly those separable Banach spaces whose dual have C^*PCP.

PROOF OF THEOREM 2. ($\Rightarrow$) This follows directly from Theorem 1(a), since if $\|\cdot\|$ is an equivalent norm on X whose dual is strictly convex, then $\|\cdot\|$ is Gâteaux differentiable.

($\Leftarrow$) Suppose X^* has C^*PCP. We will prove that if $\|\cdot\|$ is an equivalent Gâteaux differentiable norm on X, then $\|\cdot\|$ is Fréchet differentiable on a dense set. The extension to continuous, convex, Gâteaux differentiable functions then follows by the methods of Namioka and Phelps [17, Thm 6].

Thus, let $\|\cdot\|$ be an equivalent Gâteaux-smooth norm on X. For each positive integer n, let

$$U_n = \left\{ x \in \mathscr{S}_X : \begin{matrix} \exists f_x \in \mathscr{S}_{X^*} \text{ such that } f_x(x) = 1 \text{ and } f_x \text{ has a weak*-open} \\ \text{neighbourhood } W \text{ with diam } W \cap \mathscr{B}_{X^*} < n^{-1} \end{matrix} \right\}.$$

We will show that U_n is norm open and dense in $\mathscr{S}_X$, from which the desired result follows by the same methods as used in Theorem 1.

The fact that U_n is open is obtained from the Gâteaux differentiability of the norm, since then the mapping $x \mapsto f_x$ is point-valued and norm-to-weak* continuous.

For density, let $0 < \varepsilon < \frac{1}{2}$, $x_0 \in \mathscr{S}_X$, and, as in Lemma 4, let $C = \operatorname{co}((x_0^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_{X^*}) \cup \mathscr{B}_{X^*})$, where we consider $x_0 \in X^{**}$. Then, just as in Lemma 4, C is weak*-compact, and there is an $f_0 \in \mathscr{B}_{X^*}$ which is a point of weak*-to-norm continuity for C, with $f_0(x_0) > 0$. Let V be an elementary weak*-open neighbourhood of f_0 such that diam $V \cap C < (2n)^{-1}$. We may clearly assume that $g(x_0) > 1/2$ if $g \in V \cap C$. By the Bishop–Phelps Theorem, there is a $g \in V \cap C$ and and an $x_1 \in \mathscr{S}_X$ such that x_1 supports C at g. By the definition of C, there are $f_1 \in \mathscr{B}_{X^*}$, $f_2 \in x_0^{-1}(0) \cap \varepsilon^{-1}\mathscr{B}_{X^*}$ and $0 < t \leqq 1$ such that $g = tf_1 + (1-t)f_2$. By the choice of V, $t > 1/2$. Thus

$$W = \frac{1}{t} V - \frac{1-t}{t} f_2$$

is a weak*-open neighbourhood of f_1 with diam $W \cap C < n^{-1}$, and, by convexity, x_1 supports $\mathscr{B}_{X^*}$ at f_1. Thus $x_1 \in U_n$. Also, x_0 and x_1 satisfy the hypotheses of Lemma 5. Since $f_1(x_1) = 1$ and $f_1(x_0) > 0$, we have, by Lemma 5, that $\| x_1 - x_0 \| < 2\varepsilon$. Thus, U_n is dense. ■

COROLLARY 8. *Let X be a separable Banach space. The following are equivalent:*

(a) *X^* has C^*PCP.*

(b) *The dual unit ball to every equivalent Gâteaux differentiable norm on X is weak*-dentable.*

(c) *The dual unit ball to every equivalent Gâteaux differentiable norm on X is the weak*-closed convex hull of its weak*-strongly exposed points.*

REMARKS. (a) Suppose that X is a separable Phelps space which is not an Asplund space, and let $\| \cdot \|$ be an equivalent Gâteaux differentiable norm on X. Let $\mathcal{D} : \mathcal{S}_X \to \mathcal{S}_{X^*}$ be the duality mapping. Then $\mathcal{D}$ is norm-to-weak* continuous, and, by the Bishop–Phelps Theorem, has norm dense range. By Theorem 1, $\mathcal{D}$ is norm-to-norm continuous at the points of a norm dense $\mathcal{G}_\delta$, say G, of $\mathcal{S}_X$. Hence $\mathcal{D}(G)$ is separable and weak*-dense in $\mathcal{S}_{X^*}$, and consists of points of weak*-to-norm continuity of $\mathcal{B}_{X^*}$. Since X^* is not separable, $\mathcal{D}$ cannot be the pointwise limit of a sequence of norm-to-norm continuous functions. Note also that a norm with these properties has the *Weak Compact Intersection of Balls Property* [21].

(b) Again motivated by the *RNP*-Asplund duality mentioned in the Introduction, we define a dual Banach space X^* to be a *weak*-Phelps space* if every continuous, convex, Gâteaux differentiable dual function on X^* is Fréchet differentiable on a dense set. Then, if X^* is separable, the methods used to prove Theorems 1 and 2 suffice to show that *X^* is a weak*-Phelps space if and only if X has CPCP*. (If the sequence $\{ f_j \}$ of the proof of Theorem 1 is chosen to be norm dense in $\mathcal{B}_{X^*}$, then the norm $\| \cdot \|$ constructed has Gâteaux differentiable dual. See, for example, [22].)

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